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A Note on Finite Groups Whose Power Graph is a Commuting Graph

Alireza Doostabadi

Faculty of Sciences, University of Zabol, Zabol, Iran

Corresponding author's e-mail: *aldoostabadi@uoz.ac.ir*

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Abstract

Let G be a finite group. The power graph of the group G, denoted by $\mathcal{P}(G)$, is a graph such that its vertex set is the group G and two distinct elements x,y are adjacent if and only if $x = y^n$ or $y = x^n$ for some positive integer n. The purpose of this paper is to study finite groups such that their commuting graph is a power graph.

1. Introduction

The investigation of algebraic structures using the properties of graphs is an important topic for some researchers. The different types of graphs with respect to groups are defined as: Cayley graphs, Commuting graphs, and Power graphs. The power graph $\mathcal{P}(G)$ of a group G, is the graph whose vertex set is the group G such that two distinct elements are adjacent if one is a power of the other.

For the first time, Kelarev and Quinn [1] have studied the directed power graph of semigroups, in which there is an arc from a vertex x to a vertex y if y is positive power of x. Cameron and Ghosh [2] have proved that two finite Abelian groups are isomorphic if and only if they have isomorphic power graphs. Some graph theoretical properties of power graphs including planarity and perfectness were discussed by Doostabadi et al. [3]. They have obtained the independence and chromatic numbers of power graphs of cyclic and arbitrary groups too [3]. Suppose that G is a finite group with power graph $\mathcal{P}(G)$. We know that if the elements x,y ∈ G are adjacent in the $\mathcal{P}(G)$, then $xy = yx$. Thus the commutativity of the elements x,y is a necessary condition for x is adjacent to y in the graph $\mathcal{P}(G)$. By using the commutativity of the elements of a group, we can define a commuting graph. More precisely, for a nonempety set X of G, the commuting graph $C(G, X)$ is the graph with X as the vertex set and two distinct elements of X being joined by an edge if they are commuting elements of G. This type of graphs has been studied for a wide variety of groups G and a selection of subsets of G. For example,commuting graphs when G is a symmetric group, have been investigated by Bates et al. [4] and Bundy [5]. Some of numerical parameters of commuting graph are stated by Mahmoudifar and Moghaddamfar in [6]. In this paper, we study groups that the necessary and sufficient condition for adjacency elements in the $\mathcal{P}(G)$ is commutative. In other words, $\mathcal{P}(G) = C(G, G)$. We use the notation $\mathcal{P}_c(G)$ for the finite groups with this property.

2. Results and Discussion

In this section, we first state some of preliminary results on the finite p-groups, and then we characterize finite groups for which their commuting graph is equals to a power graph, in Theorem 2.5.

2.1 *Theorem*

Let G be a finite p-group with the graph $\mathcal{P}_c(G)$ where p is prime. Then the group G is cyclic or generalized quaternion.

Proof. Suppose that G is a finite p-group with the graph $\mathcal{P}_c(G)$. Let z be in $Z(G)$ of prime order p. Since $[x, z] = 1$ for every element x of order p, then $\langle z\rangle = \langle x\rangle$. Hence the finite p-group G has a unique cyclic subgroup of order p. By (5.3.6) of [7], G is cyclic or generalized quaternion.

2.2 *Lemma*

If G is a finite group with the graph $\mathcal{P}_c(G)$. Then, the elements of G are p-element for some prime number p.

Proof. By contradiction, assume that there exists $x \in G$ such that pq divides order x, where p, q are distinct primes. Let $y, z \in \langle x\rangle$ of orders p and q. It is clearly, $[y, z] = 1$, but y is not adjacent z in the graph $\mathcal{P}_c(G)$, which is a contradiction.

2.3 *Corollary*

Suppose that G is a finite group with the graph $\mathcal{P}_c(G)$. Then, centralizer nontrivial elements is p-group for some prime number p. Particularly if G is not p-group, then $Z(G)$ is trivial.

We define the class CP of finite groups in which the centralizers of all nontrivial elements contain only elements of prime power order. By the previous lemma, The finite groups with the graph $\mathcal{P}_c(G)$ are in the class CP-groups. In the next theorem, Deaconescu characterized CP-groups [8].

2.4 *Theorem*

A group G is a CP-group if and only if one of the following properties hold:

(1) G is isomorphic with $PSL(2, q)$ with q = 4,7,8,9,17; $PSL(3,4)$, $SZ(8)$, $SZ(32)$ or M_{10}.

(2) G has a nontrivial normal 2-subgroup P and $\frac{G}{P}$ is isomorphic with $PSL(2,4)$, $PSL(2,8)$, $SZ(8)$ or $SZ(32)$. Moreover, P is elementary abelian and isomorphic with a direct sum of natural modules.

(3) G is a p-group.

(4) G is a frobenius group whose kernel is a p-group and the complement is either a cyclic q-group ($q \neq p$) or a generalized quaternion group.

(5) G is a 3-step group of order $p^a q^b$ (p, q primes, $q > 2$), that is,. $G = O_{pp'p}(G)$ and $G \supset O_{p'p}(G)$ with the following properties:

(i) $O_{p'p}(G)$ is a Frobenius group with kernel $O_p(G)$ and cyclic complement.

(ii) $\frac{G}{O_p(G)}$ is a Frobenius group with kernel $\frac{O_{pp'}(G)}{O_p(G)}$.

2.5 *Theorem*

Let G be a finite group with the graph $\mathcal{P}_c(G)$. Then G is isomorphic with one of the following groups:

(1) The group H_p that is a cyclic p-group or generalized quaternion group for some prime number p,

(2) $H_p \rtimes H_q$

(3) $H_p \rtimes (H_q \rtimes H_p)$.

where p and q are distinct primes.

Proof. If G is a finite p-group, then by Theorem 2.1, G is given in the part (1). Suppose that G is not a p-group. Whether G is a group with the mentioned property, by Theorem 2.1, all Sylow p-subgroups of G are cyclic or generalized quaternion. If all Sylow p-subgroups of G are cyclic, then groups G' and $\frac{G}{G'}$ are cyclic and they have coprime orders, by Theorem (5.16) of [9]. On the other hand, these groups are p-groups by Lemma 2.2. Hence G is a Frobenius group and $G \cong H_p \rtimes H_q$, where H_p is cyclic. Suppose that all Sylow subgroups of odd order are cyclic and the Sylow 2-subgroup is a generalized quaternion group. If G is not solvable, then G contains a normal subgroup G_1 with the properties that $[G: G_1] \leq 2$ and G_1 is a direct product of a Z-group (A group that is all Sylow subgroups are cyclic) and a subgroup isomorphic with $SL(2, p)$ for some odd prime number p. Indeed $SL(2, p)$ has some element of order $2p$, which is a contradiction. Hence assume that G is solvable. Since G is a CP-group, G is not isomorphic to groups of (1) and (2) in the Theorem 2.4. Thus G is isomorphic to one of the groups (4) and (5), and the proof is complete.

3. Conclusions

For a finite group G, it is clear that the power graph of G is a spanning subgraph of the commuting graph. In Theorem 2.5, the structure of groups with property $\mathcal{P}_c(G)$ is satisfied. For example, $(S3) = C(S_3, S_3)$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding this article.

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