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State KU-Algebras

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Abstract

In this paper, we introduce the concept of state on KU-algebras and prove some of their properties. Also, we analyze the relationship of their mapping with KU-substructures.

1. Introduction

Logical algebras have become the keen interest for researchers in recent years and intensively studied under the influence of different mathematical concepts. KU-algebra is a new algebraic structure introduced by Prabpayak and Leerawat [1]. Mostafa et al. in 2011 studied KU-algebra in fuzzy context and studied fuzzy KU-ideals of KU-algebras [2]. Recently, Ansari and Koam [3] introduced the concept of roughness in KU-algebras. Koam et al. [4] introduced a pseudo-metric on KU-algebras. Senapati and Shum defined Atanassov’s intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra [5]. Flaminio and Montagna were the first to present a unified approach to state and probabilistic many-valued logic in a logical and algebraic setting [6]. They added a unary operation, called internal state or state operator to the language of MV-algebras, which preserves the usual properties of states. State on BL-algebras was introduced and investigated by Iampan [7]. The state on BCK-algebras was defined and studied by Mostafa et al. in 2015 [8]. In Section 2, some basic definitions are presented. In Section 3, the state KU-algebra is defined. Some basic examples and theorems will be presented.

The λ-commutative KU-algebras are determined, and it will be shown that if (A, λ) is a λ-commutative KU-algebra in which λ (A) is closed under ∨ and ∧, then λ (A) is a distributive lattice with respect to the operations ∨ and ∧.

2. Definitions and Preliminaries

2.1 ***Definition***

[7] An algebra (A; ∗, 0) of type (2, 0) is a KU-algebra if for all x, y, z ∈ A, the following conditions hold:

(KU1) (y ∗ x) ∗ ((x ∗ z) ∗ (y ∗ z)) = 0.

(KU2)0∗x=x.

(KU3)x∗0=0.

(KU4) If x ∗ y = 0 and y ∗ x = 0, then x = y. A binary relation ⩽ in A is defined by x ⩽ y if and only if x ∗ y = 0, for all x, y ∈ A.

2.2 *Theorem*

[7] An algebra (A, . ,0) of type (2, 0) is a KU-algebra if and only if for all x, y, z ∈ A, the following conditions hold:

(1) (y . x).((x . z).(y . z)) = 0.

(2) x ((x. y).y) = 0.

(3) x. x = 0.

(4) x.0 =0.

(5) If x. y = 0 and y. x = 0, then x = y.

2.3 *Example*

[8] Let $X = \{1, 2, 3, 4, 5\}$, and let ∘ be defined as the following table:

∘	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

It is easy to see that X is a KU-algebra.

In what follows, let $(X, 0, 1)$ denote a KU-algebra, unless otherwise specified. For brevity, we also call X a KU-algebra. The element 1 of X is called constant which is the fixed element of X.

Partial order $\leq$ in X is defined by $x \leq y$ if and only if $y \circ x = 1$.

2.4 *Proposition*

[7] If (A; ∗, 0) is a KU-algebra, then the following properties hold:

(5) For any x, y, z ∈ A; z ∗ (y ∗ x) = y ∗ (z ∗ x).

(6) For any x, y ∈ A; x ∗ ((x ∗ y) ∗ y) = 0.

(7) For any x, y ∈ A; (x ∗ y) ∗ [((x ∗ y) ∗ y) ∗ y)] = 0.

(8) For any x, y, z ∈ A; x ⩽ x.

(9) If x ⩽ y and y ⩽ x, then x = y.

(10) If x ⩽ y and y ⩽ z, then x ⩽ z.

(11) If x ⩽ y, then z ∗ x ⩽ z ∗ y.

(12) If $x \leqslant y$, then $y * z \leqslant x * z$.

(13) For any $x, y \in A$; $x \leqslant y * x$.

(14) For any $x, y \in A$; $x \leqslant y * y$.

2.5 *Definition*

A nonempty subset Y of a KU-algebra K is called a subalgebra of K if for all $x, y \in Y$, we have $x \circ y \in Y$.

A KU-algebra K is said to be commutative if for all $x, y \in K$, we have $(x \circ y) \circ y = (y \circ x) \circ x$.

2.6 *Definition*

Let $(A; *, 0)$ be a KU-algebra. A nonempty subset I of A is called an ideal if

(1) $0 \in I$,

(2) $x * y \in I$ for any $x, y \in A$.

2.7 *Definition*

An ideal I of a KU-algebra K is called a commutative ideal if for all $x, y \in I$ we have $(y \circ x) \in I \Rightarrow ((x \circ y) \circ y) \circ x. \in I$.

2.8 *Example*

([3]): Let $X = \{1, 2, 3, 4, 5, 6\}$ and let $\circ$ be defined as follows:

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	1	1	1	2	1	6
6	1	1	2	1	1	1

Clearly $(X, 0, 1)$ is a KU-algebra. It is easy to show that A = {1, 2} and B = {1, 2, 3, 4, 5} are KU-ideals of X.

2.9 *Definition*

Let $(A; *, 0)$ be a KU-algebra.

(i) The KU-algebra A is self-distributive if $x * (y * z) = (x * y) * (x * z)$ for any $x, y, z \in A$.

(ii) The KU-algebra A is a commutative if $(x * y)*y=(y*x)*x$ for any $x, y, z \in A$.

3. Results and Discussion

3.1 *Definition*

Let $(A; *, 0)$ be a KU-algebra. A mapping $\lambda: A \to A$ is called a state operator on $(A; *, 0)$ if it satisfies the following properties for all $x, y, z \in A$;

(SO1) $x * y = 0$ implies $\lambda(x) * \lambda(y)=0$;

(SO2) $\lambda(x * y) = \lambda\, ((x * y) * y)\, \lambda(y)$;

(SO3) $\lambda(\lambda(x) * \lambda(y)) = \lambda(x) * \lambda(y)$.

3.2 *Lemma*

Let (A, λ) be a stat KU-algebra. Then

(1) $\lambda(0) = 0$;

(2) $\lambda(\lambda(x)) = \lambda(x)$;

(3) $\lambda(x * y) \leqslant \lambda(x) * \lambda(y)$;

(4) $\mathrm{Ker}(\lambda) = \lambda^{-1}(\{0\})$ is an ideal.

(5) $\lambda(X) = \{\lambda(x) \mid x \in X\}$ is a subalgebra of X.

(6) $\mathrm{Ker}(\lambda)\ \mathrm{Im}(\lambda) = \{0\}$.

3.3 *Example*

Let X be a nonempty set. Define a binary operation $*$ on the P(X), the power set of X, by putting $A * B = B \backslash A$ for all A, B $\in$ P(X). Then $(P(X); *, \emptyset)$ is a KU-algebra. The mapping $Id_{P(X)} : P(X) \to P(X)$ defined by $Id_{P(X)}(A) = A$ for all set $A \subset X$ is a state operator on P(X).

Proof. **SO1** and **SO3** are trivial.

SO2. Note that

$$\begin{aligned}\lambda((A * B) * B) * \lambda(B) &= B \backslash (B \backslash [B \backslash A]) \\ &= B \backslash A = A * B = \lambda(A * B).\end{aligned}$$

3.4 *Example*

Let $(A; *, 0)$ be a KU-algebra. Then the mapping $Id_X : X \to X$ defined by $Id_X(x) = x$ for all $x \in X$ is a state operator on X.

Proof. **SO1** and **SO3** are trivial.

SO2. Note that by proposition 2.3 for all $x, y \in X$, we have $x \leqslant (x * y)$. Therefore $((x * y) * y) \leqslant (x * y)$. On the other hand, $(x * y) * [((x * y) * y) * y] = ((x * y) * y) * ((x * y) * y) = 0$.

Thus $x * y \leqslant ((x * y) * y) * y$. Therefore, $x * y = ((x * y) * y) * y$.

3.5 *Lemma*

Let (A, λ) be a state commutative KU-algebra. For all $x, y \in A$, if $x \leqslant y$, then $\lambda(y * x) = \lambda(y) * \lambda(x)$.

Proof. Note that

$$\begin{aligned}(y * x) &= \lambda((y * x) * x) * \lambda(x) \\ &= \lambda((x * y) * y) * \lambda(x) = \lambda(0 * y) * \lambda(x) = \lambda(y) * \lambda(x).\end{aligned}$$

3.6 *Definition*

A state KU-algebra (A, λ) is called

(i) λ- Commutative if $(\lambda(x) * \lambda(y)) * \lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$,

(ii) λ- Distributive if $\lambda(x) * (\lambda(y) * \lambda(z)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$,

(iii) λ- Transitive if $(\lambda(y) * \lambda(z)) \leqslant (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$,

for any $x, y, z \in A$.

It is easy to see that a state KU-algebra (A, λ) is λ-commutative if $\lambda(x) = (\lambda(x) * \lambda(y)) * \lambda(x)$ for any $x, y \in A$.

3.7 ***Proposition***

Every λ-commutative state KU-algebra is λ-transitive.

Proof. Let (A, λ) be a λ-commutative state KU-algebra. Let $x, y, z \in A$. If set $\alpha = \lambda(x)$, $\beta = \lambda(y)$, and $\gamma = \lambda(z)$, then

$$\begin{aligned}(\beta * \gamma) * ((\alpha * \beta) * (\alpha * \gamma)) &= (\alpha * \beta) * ((\beta * \gamma) * (\alpha * \gamma)) = (\alpha * \beta) * (\alpha * ((\beta * \gamma) * \gamma)) \\ &= (\alpha * \beta) * (\alpha * ((\gamma * \beta) * \beta)) = (\alpha * \beta) * ((\gamma * \beta) * (\alpha * \beta)) \\ &= (\gamma * \beta) * ((\alpha * \beta) * (\alpha * \beta)) = (\gamma * \beta) * 0 = 0.\end{aligned}$$

Since $\lambda(y) * \lambda(z) \leqslant (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$, thus (A, λ) is λ-transitive.

3.8 ***Proposition***

Every λ-distributive state KU-algebra is λ-transitive.

Proof. Let (A, λ) be a λ-distributive state KU-algebra. Then for $x, y, z \in A$.

$\lambda(y) * \lambda(z) \leqslant \lambda(x) * (\lambda(y) * \lambda(z)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$.

Hence, (A, λ) is λ-transitive.

3.9 ***Proposition***

A state KU-algebra (A, λ) is λ-commutative if and only if for all $x, y \in A$,

$(\lambda(x) * \lambda(y)) * \lambda(y) \leqslant (\lambda(y) * \lambda(x)) * \lambda(x)$.

Proof. It is obvious that this condition is necessary. Conversely, let $x, y \in A$,

$(\lambda(x) * \lambda(y)) * \lambda(y) \leqslant (\lambda(y) * \lambda(x)) * \lambda(x)$. By interchanging x and y, we have

$(\lambda(y) * \lambda(x)) * \lambda(x) \leqslant (\lambda(x) * \lambda(y)) * \lambda(y)$. Hence for all $x, y \in X$, $(\lambda(x) * \lambda(y)) * \lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$.

Therefore (X, λ) is a λ-commutative state KU-algebra.

3.10 ***Remark***

Let (A, λ) be a state KU-algebra. If $x \leqslant y$, then $\lambda(z) * \lambda(x) \leqslant \lambda(z) * \lambda(y)$ and $\lambda(y) * \lambda(z) \leqslant \lambda(x) * \lambda(z)$ for all $x, y, z \in A$. Thus A is called ordered KU-algebra with respect to λ.

3.11 ***Remark***

Let A be a KU-algebra and let λ be a state operator on A. If A is self-distributive (commutative), then (A, λ) is λ-distributive (λ-commutative).

3.12 ***Proposition***

Let (A, λ) be a state KU-algebra. If λ is surjective and (A, λ) is λ-distributive (λ-commutative), then A is self-distributive (commutative) KU-algebra.

3.13 ***Lemma***

Let (A, λ) be a λ-transitive KU-algebra. Then for all $x, y, z \in A$, it holds that

$\lambda(x) * \lambda(y) \leqslant (\lambda(y) * \lambda(z)) * (\lambda(x) * \lambda(z))$.

Proof. Note that

$$[\lambda(x) * \lambda(y)] * [(\lambda(y) * \lambda(z)) * (\lambda(x) * \lambda(z))] = [\lambda(y) * \lambda(z)] * [(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))]$$
$$= [\lambda(y) * \lambda(z)] * [(\lambda(x) * (\lambda(y)) * \lambda(z))]$$
$$= \lambda(x) * [(\lambda(y) * \lambda(z)) * (\lambda(x) * \lambda(z))]$$
$$= \lambda(x) * 0 = 0.$$

Thus we have $\lambda(x) * \lambda(y) \leqslant (\lambda(y) * \lambda(z)) * (\lambda(x) * \lambda(z))$.

3.14 ***Proposition***

Let (A, λ) be a state KU-algebra with respect to λ. Then for any $x, y, z \in A$, it holds that

$$(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)) \leqslant \lambda(x) * (\lambda(y) * \lambda(z))$$

Proof. Let $x, y, z \in A$. Then $\lambda(y) \leqslant \lambda(x) * \lambda(y)$. Hence $(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)) \leqslant \lambda(y) * (\lambda(x) * \lambda(z))$. Therefore $(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)) \leqslant \lambda(x) * (\lambda(y) * \lambda(z))$.

3.15 ***Proposition***

Let (A, λ) be a state KU-algebra. Then for all $x, y \in A$, it holds that

$((\lambda(x) * \lambda(y)) * \lambda(y)) * \lambda(y) = \lambda(x) * \lambda(y)$.

Proof. For any x, y, consider

$\lambda(x) * ((\lambda(x) * \lambda(y)) * \lambda(y)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(y)) = 0$.

Hence $\lambda(x) \leqslant (\lambda(x) * \lambda(y)) * \lambda(y)$. By Remark 3.10, we have $[(\lambda(x) * \lambda(y) * \lambda(y))] * (\lambda(y) \leqslant \lambda(x) * \lambda(y)$.

Again, consider

$(\lambda(x) * \lambda(y) * [((\lambda(x) * \lambda(y)). \lambda(y)) * (\lambda(y)] * (\lambda(y) = ((\lambda(x) * \lambda(y)) * \lambda(y)) * ((\lambda(x) * \lambda(y)) * \lambda(y)) = 0$.

Hence $(\lambda(x) * \lambda(y)) \leqslant ((\lambda(x) * \lambda(y)).\lambda(y)) * \lambda(y)$. Therefore, $(\lambda(x) * \lambda(y)) = ((\lambda(x) * \lambda(y)) * \lambda(y)) * \lambda(y)$, for all $x, y \in A$.

3.16 ***Theorem***

A λ-commutative KU-algebra is λ-distributive if and only if it satisfies the following condition for all $x, y \in A$:

$\lambda(x) * (\lambda(x) * \lambda(y)) = \lambda(x) * \lambda(y)$.

Proof. Let (A, λ) be a λ-commutative state KU-algebra. Let $x, y, z \in A$. By the proof of Proposition 3.7, we have

$\lambda(y) * \lambda(z) \leqslant (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$. Hence we get that

$$\lambda(x) * (\lambda(y) * \lambda(z)) \leqslant \lambda(x) * ((\lambda(x) * \lambda(y)) * (\lambda(x).\lambda(z)))$$
$$= (\lambda(x) * \lambda(y)) * (\lambda(x) * (\lambda(x) * \lambda(z))) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)).$$

On the other hand, by treating $a = \lambda(x)$; $b = \lambda(y)$ and $c = \lambda(z)$, we also have

$$((a * b) * (a * c)) * (a * (b * c)) = ((a * b) * (a * c)) * (b * (a * c)) = b * (((a * b) * (a * c)) * (a * c))$$
$$= b * (((a * c) * (a * b)) * (a * b)) = ((a * c) * (a * b)) * (b * (a * b))$$
$$= ((a * c) * (a * b)) * (a * (b * b)) = ((a * c) * (a * b)) * (a * 0)$$
$$= ((a * c) * (a * b)) * 0 = 0.$$

It follows that $(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)) \leqslant \lambda(x) * (\lambda(y) * \lambda(z))$, and hence

$(\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z)) = \lambda(x) * (\lambda(y) * \lambda(z))$.

Therefore, (X, λ) is λ-distributive. Conversely, assume that (A, λ) is λ-distributive. Then

$\lambda(x) * (\lambda(y) * \lambda(z)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(z))$, for all x, y, z ∈ A.

Putting x = y, it yields that

$\lambda(x) * (\lambda(x) * \lambda(y)) = (\lambda(x) * \lambda(x)) * (\lambda(x) * \lambda(y)) = 0 * (\lambda(x) * \lambda(y)) = \lambda(x) * \lambda(y)$.

3.17 *Example*

The Example 3.3 is a λ-implicative KU-algebra.

Note that $(\lambda(A) * \lambda(B)) * \lambda(A) = A \setminus (B \setminus A) = A = \lambda(A)$.

Thus (P(X), $Id_{P(X)}$) is λ-implicative.

3.18 *Proposition*

Every λ-distributive and λ-commutative state KU-algebra is λ-implicative.

Proof. Let (A, λ) be a λ-distributive and λ-commutative state KU-algebra. Let x, y, z ∈ A. Clearly we have $\lambda(x) \leqslant (\lambda(x) * \lambda(y))\, \lambda(x)$. Again, we get the following equalities:

$$((\lambda(x) * \lambda(y)) * \lambda(x)) * \lambda(x) = (\lambda(x) * (\lambda(x) * \lambda(y))) * (\lambda(x).\lambda(y)) = ((\lambda(x) * \lambda(x)) * (\lambda(x) * \lambda(y))) * (\lambda(x).\lambda(y))$$
$$= (0. (\lambda(x) * \lambda(y))) * (\lambda(x) * \lambda(y)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(y)) = 0.$$

Hence $((\lambda(x) * \lambda(y)) * \lambda(x)) \leqslant \lambda(x)$. Thus $((\lambda(x) * \lambda(y)) * \lambda(x)) = \lambda(x)$.

Therefore, (A, λ) is a λ-implicative state KU-algebra.

3.19 *Proposition*

Every λ-implicative state KU-algebra is λ-commutative.

Proof. Let (X, λ) be a λ-implicative state KU-algebra and let x, y ∈ A. Since (A, λ) is λ-implicative,

$(\lambda(y) * \lambda(x)) * \lambda(y) = \lambda(y)$. Hence $\lambda(x) * ((\lambda(x) * \lambda(y)) * \lambda(y)) = (\lambda(x) * \lambda(y)) * (\lambda(x) * \lambda(y)) = 0$,

which implies that $\lambda(x) \leqslant ((\lambda(x) * \lambda(y)) * \lambda(y))$. Also, we have

$$(\lambda(y) * \lambda(x)) * \lambda(x) \leqslant (\lambda(y) * \lambda(x)) * ((\lambda(x) * \lambda(y)) * \lambda(y)) = (\lambda(x) * \lambda(x)) * ((\lambda(y) * \lambda(x)) * \lambda(y))$$
$$= 0. ((\lambda(y) * \lambda(x)) * \lambda(y)) = (\lambda(y) * \lambda(x)) * \lambda(y).$$

Thus $(\lambda(y) * \lambda(x)) * \lambda(x) \leqslant (\lambda(y) * \lambda(x)) * \lambda(y)$. Interchanging x and y, we get $(\lambda(x) * \lambda(y)) * \lambda(y) \leqslant (\lambda(y) * \lambda(x)) * \lambda(x)$.

Thus $(\lambda(x) * \lambda(y)) * \lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$ for all x, y ∈ A. Therefore (A, λ) is a λ-commutative KU-algebra.

3.20 *Theorem*

Suppose (A, λ) is a state KU-algebra with respect to λ. Then for all x, y ∈ A, the following conditions are equivalent:

(1) (A, λ) is λ-commutative;

(2) $x \leqslant y$ implies $\lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$;

(3) $(\lambda(y) * \lambda(x)) * \lambda(x) = (((\lambda(y) * \lambda(x)) * \lambda(x))) * \lambda(y)) * \lambda(y)$.

Proof. (1) ⇒ (2): Assume that (A, λ) is λ-commutative. Suppose $x \leqslant y$. Then $\lambda(x) * \lambda(y) = 0$. Hence we have

$\lambda(y) = 0 * \lambda(y) = (\lambda(x) * \lambda(y)) * \lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$.

(2) ⇒ (3): Assume the condition (2). Since $\lambda(y) \leqslant (\lambda(y) * \lambda(x)) * \lambda(x)$, by condition (2), we have

$(\lambda(y) * \lambda(x)) * \lambda(x) = (((\lambda(y) * \lambda(x)) * \lambda(x)) * \lambda(y)) * \lambda(y)$.

(3) ⇒ (1): Assume condition (3) holds. Let x, y ∈ A. It follows from condition (3) that

$(\lambda(y) * \lambda(x)) * \lambda(x) = (((\lambda(y) * \lambda(x)) * \lambda(x)) * \lambda(y)) * \lambda(y)$.

Since $\lambda(x) \leqslant (\lambda(y) * \lambda(x)) * \lambda(x)$, so

$((\lambda(y) * \lambda(x)) * \lambda(x)) * \lambda(y) \leqslant \lambda(x) *$
$\lambda(y)$.

Hence

$(\lambda(x) * \lambda(y)) * \lambda(y) \leqslant (((\lambda(y) * \lambda(x)) * \lambda(x)) * \lambda(y)) * \lambda(y) = (\lambda(y) * \lambda(x)) * \lambda(x)$.

Thus by Proposition 3.19, we get (A, λ) is λ-commutative.

3.21 *Theorem*

A state KU-algebra (A, λ) is λ-commutative if and only if the following equality holds:

For any $x, y, z \in A$, $(\lambda(z) * \lambda(x)) * (\lambda(y) * \lambda(x)) = (\lambda(x) * \lambda(z)) * (\lambda(y) * \lambda(z))$,

Proof. Assume that (A, λ) is λ-commutative. Let $x, y, z \in A$. Then

$$\begin{aligned}(\lambda(z) * \lambda(x)) * (\lambda(y) * \lambda(x)) &= \lambda(y) * ((\lambda(z) * \lambda(x)) * \lambda(x)) = \lambda(y) * ((\lambda(x) * \lambda(z)) * \lambda(z)) \\ &= (\lambda(x) * \lambda(z)) * (\lambda(y) * \lambda(z))\end{aligned}$$

Conversely, for all $x, y, \in A$, let

$(\lambda(z) * \lambda(x)) * (\lambda(y) * \lambda(x)) = (\lambda(x) * \lambda(z)) * (\lambda(y) * \lambda(z))$.

Putting $y = 0$ yields that

$(\lambda(z) * \lambda(x)) * (\lambda(0) * \lambda(x)) = (\lambda(x) * \lambda(z)) * (\lambda(0) * \lambda(z))$. Hence $(\lambda(z) * \lambda(x)) * (0 * \lambda(x)) = (\lambda(x) * \lambda(z)) * (0 * \lambda(z))$.

Therefore $(\lambda(z) * \lambda(x)) * \lambda(x) = (\lambda(x) * \lambda(z)) * \lambda(z)$. As a result, (A, λ) is λ-commutative.

3.22 *Theorem*

Let (A, λ) be a state KU-algebra. Suppose that $\lambda(A)$ is closed with respect to $\vee$. (A, λ) is λ-commutative if and only if $(\lambda(A), \leqslant)$ is an upper semi lattice with $\vee$.

Proof. Since $\lambda(b) \leqslant (\lambda(b) * \lambda(a)) * \lambda(a)$ and $\lambda(a) \leqslant (\lambda(a) * \lambda(b)) * \lambda(b)$, we have $(\lambda(b) * \lambda(a)) * \lambda(a)$ is an upper bound of $\lambda(a)$ and $\lambda(b)$ for any $a, b \in A$. Let $\lambda(c)$ be any upper bound of $\lambda(a)$ and $\lambda(b)$. Since $\lambda(a) \leqslant \lambda(c)$, we get

$\lambda(c) = 0 * \lambda(c) = \lambda(0) * \lambda(c) = (\lambda(a) * \lambda(c)) * \lambda(c) = (\lambda(c) * \lambda(a)) * \lambda(a)$.

Also, $\lambda(b) \leqslant \lambda(c)$. We obtain $(\lambda(b) * \lambda(a)) * \lambda(a) \leqslant (\lambda(c) * \lambda(a)) = \lambda(c)$.

Hence $(\lambda(b) * \lambda(a)) * \lambda(a) \leqslant \lambda(c)$ and $(\lambda(b) * \lambda(a)) * \lambda(a)$ must be least upper bound of $\lambda(a)$ and $\lambda(b)$. Conversely, assume that $(\lambda(A), \leqslant)$ is an upper bound semi-lattice satisfying $\lambda(a) \vee \lambda(b) = (\lambda(b) * \lambda(a)) * \lambda(a)$ for all $a, b \in A$;

therefore $(\lambda(b) * \lambda(a)) * \lambda(a) = \lambda(a) \vee \lambda(b) = \lambda(b) \vee \lambda(a) = (\lambda(a) * \lambda(b)) * \lambda(b)$.

Thus, (A, λ) is λ-commutative.

3.23 *Proposition*

Let (A, λ) be a λ-commutative state KU-algebra and let $\lambda(A)$ be closed with respect to $\vee$. Then the following properties hold for all $a,b,c \in A$:

(1) $\lambda(a) * (\lambda(b) \vee \lambda(c)) = (\lambda(c) * \lambda(b)) * (\lambda(a) * \lambda(b))$.

(2) If $a \leqslant b$, then $\lambda(a) \vee \lambda(b) = \lambda(b)$.

(3) If $c \leqslant a$ and $\lambda(a) * \lambda(c) \leqslant \lambda(b) * \lambda(c)$, then $\lambda(b) \leqslant \lambda(a)$.

Proof. (**1**)- Let $a, b, c \in A$. Then

$\lambda(a) * (\lambda(b) \vee \lambda(c)) = \lambda(a) * ((\lambda(c) * \lambda(b)) * \lambda(b)) = (\lambda(c) * \lambda(b)) * (\lambda(a) * \lambda(b))$.

(**2**)- Let $a \leqslant b$. Then $\lambda(a) * \lambda(b) = 0$. Hence

$\lambda(b) = 0 * \lambda(b) = (\lambda(a) * \lambda(b)) * \lambda(b) = (\lambda(b) * \lambda(a)) * \lambda(a) = \lambda(a) \vee \lambda(b)$.

(**3**)- Let $c \leqslant a$ and let $\lambda(a) * \lambda(c) \leqslant \lambda(b) * \lambda(c)$. Hence

$$\lambda(b) * \lambda(a) = \lambda(b) * (0 * \lambda(a)) = \lambda(b) * ((\lambda(c) * \lambda(a)) * \lambda(a)) = \lambda(b) * ((\lambda(a) * \lambda(c)) * \lambda(c))$$
$$= (\lambda(a) * \lambda(c)) * (\lambda(b) * \lambda(c)) = 0.$$

Hence, $\lambda(b) \leqslant \lambda(a)$.

Let (A, λ) be a λ-commutative state KU-algebra. If λ is surjective or λ is a KU-morphism, then $\lambda(x)$ is closed with respect to $\vee$. Finally, we present some condition for which a λ-commutative state KU-algebra will be a distributive lattice with respect to the operations $\vee$ and some suitable $\wedge$ operator.

3.24 ***Theorem***

Let (A, λ) be a λ-commutative state KU-algebra and let $\lambda(A)$ be closed with respect to $\vee$. If there is a lower bound $\lambda(a)$ of $\lambda(x)$ and $\lambda(y)$, then the greatest lower bound $\lambda(x) \wedge \lambda(y)$ of $\lambda(x)$ and $\lambda(y)$ exists and

$$\lambda(x) \wedge \lambda(y) = ((\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))) * \lambda(a).$$

Proof. let $\lambda(a) \leqslant \lambda(x), \lambda(y)$. Clearly $\lambda(a) \leqslant \lambda(x) \wedge \lambda(y)$. Since $\lambda(x) * \lambda(a) \leqslant (\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))$, we have

$$[(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))] * \lambda(a) \leqslant (\lambda(x) * \lambda(a)) * \lambda(a) = \lambda(a) \vee \lambda(x) = \lambda(x).$$

Similarly, we have $[(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))]. \lambda(a) \leqslant \lambda(y)$.

Hence $[(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))] * \lambda(a)$, which is a lower bound of $\lambda(x)$ and $\lambda(y)$. Suppose that $\lambda(b)$ is another lower bound of $\lambda(x)$ and $\lambda(y)$, that is, $\lambda(b) \leqslant \lambda(x), \lambda(y)$.

Hence $\lambda(x) * \lambda(a) \leqslant \lambda(b) * \lambda(a)$ and $\lambda(y) * \lambda(a) \leqslant \lambda(b) * \lambda(a)$. Thus $(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a)) \leqslant \lambda(b) * \lambda(a)$.

Therefore

$$\lambda(b) \leqslant \lambda(b) \vee \lambda(a) = \lambda(a) \vee \lambda(b)$$
$$= (\lambda(b) * \lambda(a)) * \lambda(a) \leqslant [(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))] * \lambda(a).$$

Thus $[(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))] * \lambda(a)$ is the greatest lower bound of $\lambda(x)$ and $\lambda(y)$.

Therefore, $\lambda(x) \wedge \lambda(y) = [(\lambda(x) * \lambda(a)) \vee (\lambda(y) * \lambda(a))] * \lambda(a)$.

3.25 ***Corollary***

Let (A, λ) is a λ-commutative state KU-algebra and let $\lambda(A)$ be closed with respect to $\vee$. If the lower bound exists for every two elements of $\lambda(A)$, then $(\lambda(A), \vee, \wedge)$ is a lattice.

3.26 ***Corollary***

Let (A, λ) is a λ-commutative state KU-algebra and let $\lambda(A)$ be closed with respect to $\vee$. If there exists a lower bound for $\lambda(A)$, that is, there exists $a \in X$ such that $\lambda(a) \leqslant \lambda(x)$ for all $x \in A$, then $(\lambda(A), \vee, \wedge)$ is a lattice.

3.27. ***Proposition***

Let (A, λ) be a λ-commutative state KU-algebra and let $\lambda(A)$ be closed with respect to $\vee$. Let $a, b, c \in A$. Then the following conditions holds:

(1) $(\lambda(a) \vee \lambda(b)) * \lambda(c) = (\lambda(a) * \lambda(c)) \wedge (\lambda(b) * \lambda(c))$;

(2) $(\lambda(a) \wedge \lambda(b)) * \lambda(c) = (\lambda(a) * \lambda(c)) \vee (\lambda(b) * \lambda(c))$.

Proof: it is obvious.

4. Conclusions

We introduced the concept of state on KU-algebras. Also, we analyzed the relationship of their mapping with KU-substructures. A state KU-algebra λ (A) is closed with respect to $\vee$. Moreover, (A, λ) is λ-commutative if and only if (λ (A), $\leqslant$) is an upper semi lattice with $\vee$. Finally, if there exists a lower bound for λ (A), that is, there exists a $\in$ X such that λ (a) $\leqslant$ λ(x) for all x $\in$ A, then (λ (A), $\vee$, $\wedge$) is a lattice.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this article.

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