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Maps on States Preserving von Neumann Entropy

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Abstract

Let H be a finite-dimensional complex Hilbert space. In this note, we determine all maps on $S(H)$, which is the set of all density operators on H, that preserve the von Neumann entropy of convex combinations.

1. Introduction

The theory of (linear and nonlinear) preserver problems is a very active research area in linear algebra and operator theory on which deal with maps on subsets of algebras that preserve certain sets, relations, functions, etc. They have been treated mainly in matrix theory and in operator theory.

Let H be a finite-dimensional complex Hilbert space. We denote by $S(H)$, the space of all density operators (quantum states, because they are in one-to-one correspondence with the set of states of a quantum mechanical system whose observables are self-adjoint operators on H.) on H, i.e., positive semi-definite operators with unit trace. By $P_1(H)$, we denote the set of all rank-one projections (pure states) on H. We denote by $\lambda(\rho)$ the set of all eigenvalues of $\rho \in S(H)$.

Now we are going to introduce two very important numerical quantities for information theory that in spite of appearance similarity are completely different (see [7]). The Shannon entropy of a finite probability distribution $(p_1, p_2, \dots, p_n)$ is the quantity

$$\boldsymbol{H(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i \log_2 p_i}, \tag{1.1}$$

where $0 \log_2 0 = 0$ and $1 \log_2 1 = 0$. Quantum information can be quantitatively measured by using an analogue of Shannon entropy, called the von Neumann entropy. The von Neumann entropy of $\rho \in S(H)$, $S(\rho)$ is defined by

$$\mathbf{S}(\boldsymbol{\rho}) = -\mathbf{tr}\,(\boldsymbol{\rho}\ \log \boldsymbol{\rho}). \qquad (1.2)$$

Here tr stands for usual trace functional and log denotes the logarithm with base 2. The operator $\rho \log \rho \in S(H)$ is defined by using the Spectral Theorem. If we can interpret the probabilities p_i in Eq. (1.1) as eigenvalues of the density operator ρ, then $S(\rho)$ is numerically equal to $H(p_1, p_2, \ldots, p_n)$, i.e., $S(\rho) = H(\lambda(\rho))$.

He et al. [2] described the general form of surjective maps $\varphi : S(H) \to S(H)$ such that they preserve von Neumann entropy of convex combinations, i.e.

$$\boldsymbol{S}(\boldsymbol{t\rho} + (\mathbf{1} - \boldsymbol{t})\boldsymbol{\sigma}) = \boldsymbol{S}\big(\boldsymbol{t\varphi}(\boldsymbol{\rho}) + (\mathbf{1} - \boldsymbol{t})\boldsymbol{\varphi}(\boldsymbol{\sigma})\big) \quad (\boldsymbol{\rho}, \boldsymbol{\sigma} \in \boldsymbol{S}(\boldsymbol{H})). \qquad (1.3)$$

They used the generalization of Uhlhorn's version of Wigner's theorem (see [4]) to do this. They posed that describing the general structure maps on quantum states is an open problem. In the present paper, our aim is to give a complete description of maps $\varphi : S(H) \to S(H)$ with property (1.3) We do not have the surjectivity assumption of φ that the assumption was in [2]. We employ the non-surjective version of Wigner's theorem to do this, which reads as follows. This note is a special case of Theorem 1.1 in [3] but with a different approach.

1.1 ***Theorem***

(See [1]) Let φ be a map on $P_1(H)$ that preserves the transition probability between pure states, i.e.,

$$tr\,[PQ] = tr\,[\varphi(P)\varphi(Q)] \quad (P, Q \in P_1(H)).$$

Then there exists a unitary or an anti-unitary operator U on H such that $\varphi(P) = UPU^*$.

2. Basic Lemmas and Statement of the Main Result

In the section, we state some of basic results that we need for the proof of Theorem 2.5. The following properties of the von Neumann entropy are essential for the proof of Theorem 2.5.

2.1 ***Lemma***

(Properties of the von Neumann entropy)(see Theorem 11.8 in [6].)

(i) For all $\rho \in S(H)$, $0 \le S(\rho) \le n$, and there is equality on the left if and only if ρ is a pure state, and there is equality on the right if and only if $\rho = \frac{I}{n}$.

(ii) Unitary invariance: For any unitary operator U, $S(U\rho U^*) = S(\rho)$.

The following lemma is a key to completing the proof of Theorem 2.5.

2.2 ***Lemma***

(See Lemma 2.9 in [2].) For $\rho, \sigma \in S(H)$ with $\dim H = n < \infty$, the following statements are equivalent:

(i) $\rho = \sigma$;

(ii) $S\left(t\rho + (1-t)\frac{I}{n}\right) = S(t\sigma + (1-t)\frac{I}{n})$ for arbitrary $t \in [0,1]$ and there exists a number $t_0 \in (0,1)$ such that

$$S(t_0\rho + (1 - t_0)P) = S(t_0\sigma + (1 - t_0)P) \qquad (P \in P_1(H)).$$

In the following, a fundamental concept together with an essential lemma used in this note is given.

2.3 *Definition*

Let x and y be two n-dimensional vectors on $\mathbb{R}^n$. We say that x majorized by y (or y majorizes x), denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$$

and

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$$

where $x^{\downarrow} = (x_1^{\downarrow}, \dots, x_n^{\downarrow})$ denotes the vector whose elements are rearranged in non-increasing order. For all operators $\rho, \sigma \in S(H)$, we say that ρ is majorized by σ, denoted by $\rho \prec \sigma$, if $\lambda(\rho) \prec \lambda(\sigma)$.

2.4 *Lemma*

(See Lemma 2.3 in [4].) Let $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ with $\mathrm{x_i}, \mathrm{y_i} \geq 0$. If $\mathrm{x} \prec \mathrm{y}$ and $\mathrm{H}(\mathrm{x}) = \mathrm{H}(\mathrm{y})$, then $\mathrm{x}^{\downarrow} = \mathrm{y}^{\downarrow}$.

We are now ready to state our main result as follows.

2.5 *Theorem*

Let H be a finite-dimensional complex Hilbert space with $2 < \dim \mathrm{H} < \mathrm{n}$. Assume that $\varphi : \mathrm{S}(\mathrm{H}) \rightarrow \mathrm{S}(\mathrm{H})$ is a map such that preserves von Neumann entropy of convex combinations, i.e., it satisfies

$$S(t\rho + (1-t)\sigma) = S\big(t\varphi(\rho) + (1-t)\varphi(\sigma)\big) \qquad \big(\rho, \sigma \in S(H)\big).$$

Then there exists a unitary or an anti-unitary operator U on H such that φ is of the form

$$\varphi(\rho) = U\rho U^* \qquad \big(\rho \in S(H)\big).$$

3. Results and Discussion

3.1 *Proof of the main result*

Let φ be as in the main theorem. We will set up the proof through several steps.

Step 1. $\varphi\big(P_1(H)\big) \subseteq P_1(H)$ *and* $\varphi\left(\frac{I}{n}\right) = \frac{I}{n}$.

It follows from Lemma 2.1 (i), and the fact that $S(\rho) = S(\varphi(\rho))$.

Step 2. If P and Q are different rank-one projections which are not orthogonal to each other, then the operators $\rho = tP + (1-t)Q$ and $\sigma = t\varphi(P) + (1-t)\varphi(Q)$ have the same non-zero eigenvalues. We clearly have

$$H(\lambda(\rho)) = S(\rho) = S(\sigma) = H(\lambda(\sigma)).$$

It suffices to prove that $\lambda(\rho) \prec \lambda(\sigma)$ or $\lambda(\sigma) \prec \lambda(\rho)$, because then the assertion about non-zero eigenvalues of ρ and σ follows from Lemma 2.4. As $tr(\rho) = tr(\sigma) = 1$ and $rank(\rho) = rank(\sigma) = 2$, by the Spectral decomposition (see Theorem 3.4 in [8].) ρ and σ have non-zero eigenvalues as

$\lambda(\rho) = \{\lambda, 1 - \lambda\}$and $\lambda(\sigma) = \{\alpha, 1 - \alpha\}$, respectively. Then $\alpha \geq \lambda$ or $\lambda \geq \alpha$. If $\alpha \geq \lambda$, then clearly $\lambda(\rho) \prec \lambda(\sigma)$, and if $\lambda \geq \alpha$, then $\lambda(\sigma) \prec \lambda(\rho)$, as desired.

Step 3. φ preserves the non-zero transition probability between pure states, i.e., φ has the following property:

$$tr\,[PQ] = tr\,[\varphi(P)\varphi(Q)], \qquad (P, Q \in P_1(H)).$$

For $i = 1, \ldots, n$, denote by $S_i(A)$ the ith symmetric function on the eigenvalues of operator$A$ on H. (For example, let $n = 3$ and $\lambda(A) = \{\lambda_1, \lambda_2, \lambda_3\}$. Then $S_1(A) = tr(A)$, $S_3(A) = det(A)$, here det stands for determinant of operator A, and $S_2(A) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$.) We clearly have

$$\mathbf{tr(A)^2 = tr(A^2) + 2S_2(A)} \tag{2. 1}$$

for every operator A on H. Let P and Q be different pure states that are not orthogonal to each other. By what we have just proved in Step 3, we have $S_2(tP + (1 - t)Q) = S_2(t\varphi(P) + (1 - t)\varphi(Q))$. We use (2.1) to deduce that

$$tr((tP + (1 - t)Q)^2) = tr\left(\left(t\varphi(P) + (1 - t)\varphi(Q)\right)^2\right).$$

With a simple calculation, one can conclude that $tr(PQ) = tr(\varphi(P)\,\varphi(Q))$, as desired.

Step 4. There exists a unitary or an anti-unitary operator U on H such that the restriction of φ onto $P_1(H)$ has the form

$$\varphi(P) = UPU^* \qquad (P \in P_1(H)).$$

This assertion follows from Step 3 and the non-surjective version of Wigner's theorem. Now, we complete the proof of Theorem 2.5 by a following short argument.

Define a map ψ on $S(H)$ by $\psi(\rho) = U^*\rho U$ $(\rho \in S(H))$. It is clear that $\psi : S(H) \longrightarrow S(H)$ has the same property as φ and it is the identity map on $P_1(H)$, and then apply Lemma 2.3 which gives $\rho = \psi(\rho) = U^*\varphi(\rho)U$ $(\rho \in S(H))$. Finally, we deduce

$$\varphi(\rho) = U\rho U^*, \qquad (\rho \in S(H)),$$

and the proof is complete, as desired.

4. Conclusions

In [8], authors characterized the form of surjective maps on *S(H)* preserving the von Neumann entropy and Tsallis p-entropy of a convex combination when *H* is a separable Hilbert space of any dimensions. With this introduction, the infinite-dimensional version of the theorem 2.5 can be interesting.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding this article.

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