

A note on the Independence Number of a Power Graph of a Cyclic Group

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Abstract

Let G be a finite group. The power graph of the group G , with notation $\mathcal{P}(G)$ is a graph such that its vertex set is the group G and two distinct elements x, y are adjacent if and only if $x = y^n$ or $y = x^n$ for some positive integer n . In this note, we compute bounds of independence number of power graph of a cyclic group.

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1. Introduction

There are a number of constructions of graphs from groups or semigroups in the literature. For example, the most important of them are Cayley graphs, Commuting graphs and Power graphs. The power graph $\mathcal{P}(G)$ of a group G , is the graph whose vertex set is the group G so that two distinct elements are adjacent if one is a power of the other. For the first time, Kelarev and Quinn [5] have studied the directed power graph of semigroups, in which there is an arc from a vertex x to a vertex y if y is positive power of x . Cameron and Ghosh [2] have proved that two finite Abelian groups are isomorphic if and only if they have isomorphic power graphs. Some graph theoretical properties of power graphs including planarity and perfectness have been discussed by Doostabadi et al. [4]. They have obtained chromatic numbers of power graphs of cyclic and arbitrary groups too [4]. Suppose that G is a finite group with power graph $\mathcal{P}(G)$. We know that if the elements $x, y \in G$ are adjacent in the $\mathcal{P}(G)$, then $xy = yx$. Thus commutativity of the elements x, y is necessary condition for x is adjacent to y in the graph $\mathcal{P}(G)$. In [3], we studied groups that necessary and sufficient condition for adjacency elements in the $\mathcal{P}(G)$ is commutative. We compute upper bound for the independence number of power graph of a cyclic group and in the special case, we presented a formula for the independence number of $\mathcal{P}(Z_{p^m q^n})$.

2. Results and Discussion

In this section, we will present some results about independence number of the power graph. For any finite group G the induced subgraph on $A \cup \{1\}$, where $A = \{g \in G : |g| = 3\}$, consists of $|A|/2$ triangles with the identity

element as common vertex. Hence $\alpha(G) \geq |A|/2$. Moreover, if G is a group of prime exponent p , then $P(G)$ is a graph consisting of $(|G| - 1)/(p - 1)$ complete subgraphs of order p sharing the identity vertex, which implies that $\alpha(P(G)) = (|G| - 1)/(p - 1)$. Now, we may focus on cyclic groups as the following.

2.1 Lemma

Let G be a cyclic group and $x, y \in G$. Then x, y are adjacent in the graph $p(G)$ if and only if $|x|$ divides $|y|$ or conversely.

Proof. Suppose that x, y are adjacent, then $x \in \langle y \rangle$ or $y \in \langle x \rangle$, hence $|x|$ divides $|y|$ or conversely. Now, without diminishing the wholeness of the proof, assume $|x|$ divides $|y|$. Since in the cyclic groups, the subgroups of specified order are unique, then the result is hold.

2.2 Theorem

Let G be a cyclic group of order n , where $n = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r^{\lambda_r}$, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ and p_1, p_2, \dots, p_r are distinct primes. Then

$$\alpha(p(G)) \leq (\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_{r-1} + 1).$$

Furthermore, the quality holds if and only if $\lambda_r \geq \sum_{i=1}^{r-1} \lambda_i$

Proof. First, we prove $\alpha(p(G)) \leq (\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_{r-1} + 1)$. Suppose on the contrary that S is an independent set in $P(G)$ such that $|S| \geq (\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_{r-1} + 1)$. By pigeonhole principle, there exists $x, y \in S$ such that $|x| = kp_r^s$ and $|y| = kp_r^t$ for some divisor k of $p_1^{\lambda_1} p_2^{\lambda_2} \dots p_{r-1}^{\lambda_{r-1}}$ and some $0 \leq s, t \leq \lambda_r$.

By Lemma 2.1, x, y are adjacent which is a contradiction. Now, assume that $\lambda_r \geq \sum_{i=1}^{r-1} \lambda_i$.

Put

$$U = \{p_1^{\eta_1} p_2^{\eta_2} \dots p_r^{\eta_r} \mid \sum_{i=1}^r \eta_i = \sum_{i=1}^{r-1} \lambda_i, 0 \leq \eta_i \leq \lambda_r\}.$$

Then $|U| = \prod_{i=1}^{r-1} (\lambda_i + 1)$ and U contains numbers, which are not divisible by each other. It is clear that for every $m_i \in U$ there is an element $g_i \in G$ such that $|g_i| = m_i$ for all $1 \leq i \leq |U|$. Put $U' = \{g_1, g_2, \dots, g_{|U|}\}$.

So U' is an independent set in $P(G)$ and the equality holds.

Conversely, suppose that $\alpha(p(G)) = (\lambda_1 + 1)(\lambda_2 + 1) \dots (\lambda_{r-1} + 1)$ and T is an independent set in $P(G)$ of maximum size. Let

$$T' = \{(\eta_1, \eta_2, \dots, \eta_r) : |t| = p_1^{\eta_1} p_2^{\eta_2} \dots p_r^{\eta_r} \text{ for some } t \in T\}$$

and

$$T'' = \{(\eta_1, \eta_2, \dots, \eta_{r-1}) : (\eta_1, \eta_2, \dots, \eta_r) \in T' \text{ for some } \eta_r\}.$$

since T is an independent set, we have $|T| = |T'| = |T''|$. Let

$$X = \prod_{i=1}^{r-1} \{0, 1, 2, \dots, \lambda_i\}.$$

Then, we have $T'' \subseteq X$ and $|X| = \prod_{i=1}^{r-1} (\lambda_i + 1)$, which implies that $T'' = X$. the map

$$f: T' \rightarrow X$$

$$(\eta_1, \eta_2, \dots, \eta_r) \rightarrow (\eta_1, \eta_2, \dots, \eta_{r-1})$$

is onto. Let \preceq be the lexicographic order on X and $a_1 \succ a_2 \succ \dots \succ a_k$ is maximal chain in X .

Clearly, $k = \lambda_1 + \lambda_2 + \dots + \lambda_{r-1}$. There is $r_i \in \{0, 1, 2, \dots, \lambda_r\}$ such that $f(a_i, r_i) = a_i$ for each $i = 0, 1, 2, \dots, k$, because f is onto. Since T is an independent set, we deduce $0 \leq r_0 < r_1 < r_2 < \dots < r_k \leq \lambda_r$. Hence, $\sum_{i=1}^{r-1} \lambda_i = k \leq r_k \leq \lambda_r$ and the assertion holds.

Example. $\alpha(P(Z_{p^m q^n})) = \min\{m, n\} + 1$, where p and q are distinct primes.

Proof. Suppose that $m \leq n$, by Theorem 2.2, we have $\alpha(p(G)) \leq (m + 1)$. It is sufficient we show that there exists an independent set of size $m + 1$. Let $U = \{p^m, p^{m-1}q^1, p^{m-2}q^2, p^{m-3}q^3, \dots, p^0q^m\}$.

Now, for any number u in U , we choose one element of $Z_{p^m q^n}$ with order u . Therefore, we will give an independent set of size $m + 1$.

The following theorem plays an important role in the proof of Theorem 2.4. It is known as Sperner's Theorem. (see [1])

2.3 Theorem

Let $\Delta = \{A_1, A_2, \dots, A_t\}$ be a family of nonempty subsets of $X = \{1, 2, \dots, n\}$ such that $A_i \not\subseteq A_j$

For every $1 \leq i, j \leq n$. Then

$$|\Delta| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Where $\lfloor \frac{n}{2} \rfloor$ is the integer part $\frac{n}{2}$.

2.4 Theorem

Let $n = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_r^{\lambda_r}$ be a natural number in which p_1, p_2, \dots, p_r are distinct primes and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$. Then

$$\alpha(P(Z_n)) \leq \binom{r}{\lfloor \frac{r}{2} \rfloor}$$

Where $\lfloor \frac{r}{2} \rfloor$ is the integer part $\frac{r}{2}$. Furthermore, if $r \geq 3$, then the equality holds if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_r = 1$.

Proof. Let $A = \{p_1, p_2, \dots, p_r\}$ and $P(A)$ be the power set on A . We define the map $\theta: P(A) \rightarrow S_n$, where S_n is the set of all divisors of n , by $\theta(B) = \prod_{a_i \in B} a_i$, for each $\emptyset \neq B \subseteq A$ and $\theta(\emptyset) = 1$. It is clear that $B_1 \not\subseteq B_2$ if and only if $\theta(B_1) \nmid \theta(B_2)$ for every two subsets B_1 and B_2 of A . Thus there is a one-to-one correspondence between the set of all k -element subsets of A and an independent set in $P(Z_n)$. The maximum number of k -elements subsets of A happens whenever $k = \lfloor \frac{r}{2} \rfloor$. Hence the first statement is proved.

Now, assume $\alpha(P(Z_n)) = \binom{r}{\lfloor \frac{r}{2} \rfloor}$. Suppose on the contrary that $\lambda_i > 1$ for some $1 \leq i \leq r$. For the set $S = \{\theta(B): B \in P(A), |B| = \lfloor \frac{r}{2} \rfloor\} \cup \{p_i^2\}$ we can find an independent set I such that $S = \{|x|: x \in I\}$, if $r > 3$. If $r = 3$, then we can choose the same sets S and I by replacing $|B| = \lfloor \frac{r}{2} \rfloor$ with $|B| = 2$. In the both cases, we will have an independent set of size $\lfloor \frac{r}{2} \rfloor + 1$, which is a contradiction. The convers follows directly from Spencer's Theorem.

4. Conclusions

In graph theory, an independent set is a set of vertices in a graph, no two of which are adjacent. That is, it is a set S of vertices such that for every two vertices in S , there is no edge connecting the two. A maximal independent set is an independent set that is not a proper subset of any other independent set. A maximum independent set is an independent set of largest possible size for a given graph G . This size is called the independence number of G and is usually denoted by $\alpha(G)$. The optimization problem of finding such a set is called the maximum independent set problem. It is a strongly NP-hard problem. In this paper, we compute upper bound for the independence number of power graph of a cyclic group and in the special case, we presented a formula for the independence number of $P(\mathbb{Z}_{p^m q^n})$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding this article.

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