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On Some Results of Geometric Mixture Models

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Abstract

Over recent decades, numerous methodologies have been developed to address the heterogeneity within populations. These methodologies vary in their application to both parametric and semi-parametric models, which are crucial for a broad spectrum of uses in reliability and survival analysis. Research indicates that mixture distributions serve as an effective approach to representing population heterogeneity. This study delves into geometric mixture models for survival functions (or distribution functions), exploring their inherent properties and features. We discuss various stochastic and distributional aspects of these mixtures. Additionally, we establish some conditions for stochastic comparisons based on the usual stochastic order, hazard rate order, and reversed hazard rate order. Furthermore, we integrate our findings with prominent semi-parametric models in reliability theory, including the additive hazard rate model, the proportional hazard rate model, the accelerated lifetime model, and the proportional reversed hazard rate model, which serve as foundational models in our geometric mixtures. To corroborate our findings, we will demonstrate numerical examples.

1. Introduction

It is often rare to find a homogeneous population in the real world. This may be because most real-life populations consist of a finite number of homogeneous sub-populations. Instead, heterogeneity simply occur when for instance produced products by different factories are mixed together for marketing, or they produced in the same factory due to different level of quality in work shifts, or different raw materials, etc. are combined together (Finkelstein [1]; Cha and Finkelstein [2]). Therefore, formulating and modeling heterogeneity in populations often based on some statistical distributions have been discussed. Studies show that a suitable tool for modeling heterogeneity in populations is the mixture distributions. Mixture models are usually effective tools to consider populations heterogeneity. As an application in engineering and industry, the mixture can be used to model burn-in (the process by which components of a system are tested before being put into service and will cause definitive failures to occur under controlled conditions).

There are two models to study the heterogeneity in populations: arithmetic mixture and geometric mixture models. These models provide completely different models for the study of heterogeneous components. This then raises the question which of these two mixture models is more suitable or is to be preferred to study the consequences of heterogeneity in populations. This is a fundamental question and sometimes causes disagreements between authors and researchers. For example, to model burn-in, Block and Savits [3] have used a mixture of probability distribution functions for failure time, while Lynn and Singpurwalla [4] were against this choice and have argued in favor of the predictive failure rate function, which is a mixture of failure rates. Many authors have investigated various aspects of the arithmetic mixture. Recently, Asadi et al. [5] have proposed a flexible mixture model that indexed by parameter α ∈ R, so-called α-mixture, and have studied some properties of the suggested model. The α-mixture model includes both the arithmetic mixture model and the geometric mixture model. See also Shojaee et al. [6-8] and Shojaee and Momeni [9]. For more applications of the geometric mixture, we refer to Shojaee and Babanezhad [10].

The rest of the paper is organized as follows. In section 2, some definitions and theorem that will be used in the paper are presented. In section 3, we define the geometric mixture models and provide their ageing properties. Section 4 devotes to ordering results of geometric mixture models in sense of usual stochastic order and the hazard (reversed hazard) rate order. Finally, section 5 concludes the paper.

2. Preliminaries

Let $f(.)$ and $g(.)$, $F(.)$ and $G(.)$, $\overline{F}(.)$ and $\overline{G}(.)$, $r_X(.)$ and $r_Y(.)$, $\widetilde{r}_X(.)$ and $\widetilde{r}_Y(.)$, $F^{-1}(.)$ and $G^{-1}(.)$ be the PDF's, CDF's, SF's, hazard rate functions, reversed hazard rate functions and quantile functions of random variables X and Y, respectively. The following definitions can be helpful in our derivations.

Definition (2.1) *F is said to be smaller than G in the*

- *usual stochastic order* if $\overline{F}(t) \le \overline{G}(t)$ for all t, and denoted by $F \le_{st} G$.
- *hazard rate order* if $\overline{G}(t)/\overline{F}(t)$ is increasing in t, and denoted by $F \le_{hr} G$; or equivalently $r_X(t) \ge r_Y(t)$, for all t.
- *reversed hazard rate order* if $G(t)/F(t)$ is increasing in t, and denoted by $F \le_{rh} G$; or equivalently $\widetilde{r}_X(t) \le \widetilde{r}_Y(t)$, for all t.

Definition (2.2) *(Joag-Dev et al. [11]). A pair of measurable real functions (k_1, k_2), is said to satisfy positivity of the second order determinant (DP_2) condition if*

- k_1 is nonnegative while k_2 may take negative values.
- for every $t_1 \le t_2$,

$$k_1(t_1)\, k_2(t_2) \ge k_1(t_2)\, k_2(t_1).$$

Theorem (2.3) *(Joag-Dev et al. [11]). Let (k_1, k_2) be a pair of functions satisfying the DP_2 property and the SF's $\overline{F}(t,\gamma)$ be TP_2 in (γ, t). Suppose that for $i = 1,2$, $\int k_i(t)\, dF_\gamma(t)$ exists and is finite. Furthermore, suppose that $k_l(t)$ is increasing in t. Then, for $i = 1,2$, $h_i(t) = \int k_i(t)\, dF_\gamma(t)$ is DP_2, or equivalently,*

$$\int k_1(t)\, dF_1(t) \int k_2(t)\, dF_2(t) \geq \int k_1(t)\, dF_2(t) \int k_2(t)\, dF_1(t).$$

3. Geometric mixture models and their ageing properties

Suppose that the non-negative random variable Γ is a covariate (or frailty). For mixing random variable Γ, the geometric mixture of SF's, denoted by $\overline{F}_{gm}(t)$, is defined as follows:

$$\overline{F}_{gm}(t) = \exp\left(\int_0^\infty \log \overline{F}(t \mid \gamma)\pi(\gamma)d\gamma\right), \tag{1}$$

where $\pi(\gamma)$ is the PDF of Γ. The corresponding PDF of (1) will be equal to

$$f_{gm}(t) = \left(\int_0^\infty r(t \mid \gamma)\pi(\gamma)d\gamma\right)\overline{F}_{gm}(t), \tag{2}$$

where $r(t \mid \gamma)$ is the baseline hazard rate. Thus, the hazard rate of geometric mixture of SF's, $r_{gm}(t)$, is

$$r_{gm}(t) = \frac{f_{gm}(t)}{\overline{F}_{gm}(t)} = \frac{\left(\int_0^\infty r(t \mid \gamma)\pi(\gamma)d\gamma\right)\overline{F}_{gm}(t)}{\overline{F}_{gm}(t)} \tag{3}$$

$$= \int_0^\infty r(t \mid \gamma)\pi(\gamma)d\gamma. \tag{4}$$

Relation (4) shows that if $\overline{F}(t \mid \gamma)$ is decreasing (increasing) failure rate, then $\overline{F}_{gm}(t)$ is, also, decreasing (increasing) failure rate.

Similar to the case of geometric mixture of SF's, we can define geometric mixture of CDF's, denoted by $F_{gmC}(t)$, as follows:

$$F_{gmC}(t) = \exp\left(\int_0^\infty \log F(t \mid \gamma)\pi(\gamma)d\gamma\right). \tag{5}$$

The corresponding PDF of (5) is

$$f_{gmC}(t) = \left(\int_0^\infty \tilde{r}(t \mid \gamma)\pi(\gamma)d\gamma\right)F_{gmC}(t), \tag{6}$$

where the baseline reversed hazard function is $\tilde{r}(t \mid \gamma)$. Also, the reversed hazard rate function in this case, $\tilde{r}_{gm}(t)$, is as follows:

$$\tilde{r}_{gm}(t) = \frac{f_{gmC}(t)}{F_{gmC}(t)} = \frac{\left(\int_0^\infty \tilde{r}(t \mid \gamma)\pi(\gamma)d\gamma\right)F_{gmC}(t)}{F_{gmC}(t)} \tag{7}$$

$$= \int_0^\infty \tilde{r}(t \mid \gamma)\pi(\gamma)d\gamma, \tag{8}$$

which implies that if $F(t \mid \gamma)$ is decreasing (increasing) reversed hazard rate, so $F_{gmC}(t)$.

4. Stochastic comparisons of geometric mixture models

In this section, we compare two geometric mixture of SF's (CDF's) in the sense of the usual stochastic order and the hazard rate order.

4.1 *The usual stochastic order*

The following theorem extends the result of Theorem 1.A.6 of Shaked and Shanthikumar [12] in ordinary mixture to the geometric mixture of SF's. In fact, we compare two geometric mixtures of SF's in the sense of the usual stochastic order.

Theorem (4.1) *Consider the family of SF's* $\{\overline{F}(t\mid\gamma),\gamma\in[0,\infty)\}$ *with CDF* $F(t\mid\gamma)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i, $i=1,2$, *represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i)=\exp\left(\int_0^\infty \log(\overline{F}(t\mid\gamma))d\Pi_i(\gamma)\right).$$

If $F(t\mid\gamma)\le_{st} F(t\mid\gamma')$ for $\gamma\le\gamma'$ and $\Pi_1\le_{st}\Pi_2$, then

$$F_{gm}(t,1)\le_{st} F_{gm}(t,2),$$

where $F_{gm}(t,i)$, $i=1,2$, is the corresponding CDF of $\overline{F}_{gm}(t,i)$.

Proof. From assumption $F(t\mid\gamma)\le_{st} F(t\mid\gamma')$, since $\overline{F}(t\mid\gamma)$ is increasing in γ for any t, thus $\log(\overline{F}(t\mid\gamma))$ is increasing in γ for any t. Hence, from assumption $\Pi_1\le_{st}\Pi_2$, we have

$$E_{\Gamma_1}(\log\overline{F}(t\mid\gamma))\le E_{\Gamma_2}(\log\overline{F}(t\mid\gamma)).$$

That means

$$\int_0^\infty \log(\overline{F}(t\mid\gamma))d\Pi_1(\gamma)\le\int_0^\infty \log(\overline{F}(t\mid\gamma))d\Pi_2(\gamma),$$

and consequently

$$\overline{F}_{gm}(t,1)=\exp\left(\int_0^\infty \log(\overline{F}(t\mid\gamma))d\Pi_1(\gamma)\right)\le\exp\left(\int_0^\infty \log(\overline{F}(t\mid\gamma))d\Pi_2(\gamma)\right)=\overline{F}_{gm}(t,2),$$

or $F_{gm}(t,1)\le_{st} F_{gm}(t,2)$. This completes the proof.

The following example illustrates the validity of Theorem 4.1.

Example (4.2) Consider the SF of compound Rayleigh distribution, $\overline{F}(t\mid\gamma)=(\frac{\gamma}{\gamma+t^2})^\beta$, $\gamma>0$ and $\beta>0$. It is easy to see that $\overline{F}(t\mid\gamma)$ is increasing in γ for all t, because

$$\frac{\partial}{\partial\gamma}\overline{F}(t\mid\gamma)=\beta(\frac{\gamma}{\gamma+t^2})^{\beta-1}\frac{t^2}{(\gamma+t^2)^2}\ge 0,$$

and then $T\mid\gamma$ is increasing in the sense of usual stochastic order. Now, suppose that $\Gamma: E(\theta)$, $\pi(\gamma)=\theta e^{-\theta\gamma}$. Thus

$$\overline{F}_{gm}(t,i)=\exp\left(\int_0^\infty \log((\frac{\gamma}{\gamma+t^2})^\beta)\theta_i e^{-\theta_i\gamma}d\gamma\right).$$

Set $\theta_1=3$ and $\theta_2=2$; then $\Pi_1\le_{st}\Pi_2$ and all conditions of Theorem 4.1 are satisfied. Figure 1 plots $\overline{F}_{gm}(t,i)$, $i=1,2$, for $\beta=0.9$ and $g_1(t)=\overline{F}_{gm}(t,2)-\overline{F}_{gm}(t,1)$. It is easy to see that $F_{gm}(t,1)\le_{st} F_{gm}(t,2)$.

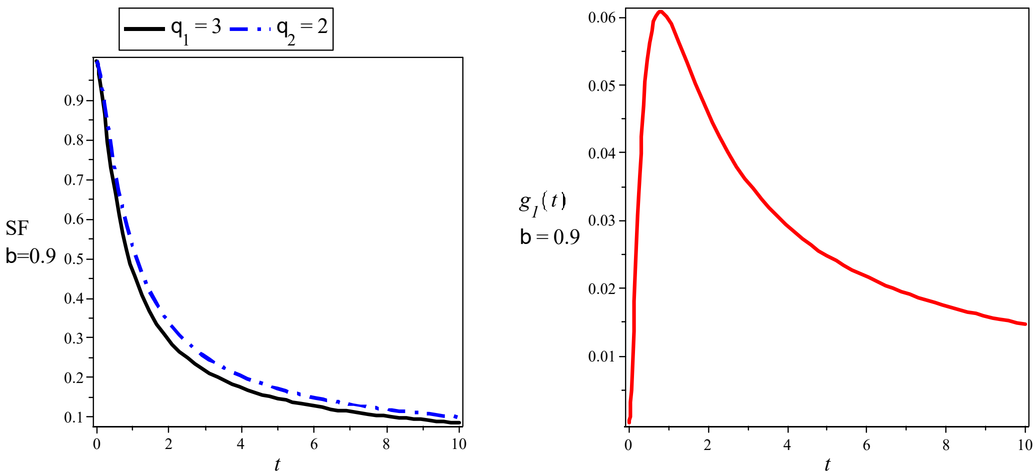

Figure 1. $\overline{F}_{gm}(t,1)$ for $\theta_1 = 3$ (solid) and $\overline{F}_{gm}(t,2)$ for $\theta_2 = 2$ (dash dot); $g_1(t) = \overline{F}_{gm}(t,2) - \overline{F}_{gm}(t,1)$ in Example 4.2 for $\beta = 0.9$

The following corollaries can be obtained directly from Theorem 4.1 in some semi-parametric families of distributions.

Corollary (4.3) *Consider the family of SF's* $\{\overline{F}(t\,|\,\gamma), \gamma \in [0,\infty)\}$, *in which the baseline SF follows from the proportional hazard rate model,* $\overline{F}(t\,|\,\gamma) = \overline{F}^{\gamma}(t)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i, $i = 1,2$, *represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i) = \exp\left(\int_0^{\infty} \gamma \log(\overline{F}(t)) d\Pi_i(\gamma)\right).$$

If $\Pi_1 \leq_{st} \Pi_2$, then

$$F_{gm}(t,1) \geq_{st} F_{gm}(t,2).$$

Corollary (4.4) *Consider the family of SF's* $\{\overline{F}(t\,|\,\gamma), \gamma \in [0,\infty)\}$, *in which the baseline SF follows from the additive hazard rate model,* $\overline{F}(t\,|\,\gamma) = \overline{F}(t)e^{-\gamma t}$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i, $i = 1,2$, *represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i) = \exp\left(\int_0^{\infty} \log(\overline{F}(t)e^{-\gamma t}) d\Pi_i(\gamma)\right).$$

If $\Pi_1 \leq_{st} \Pi_2$, then

$$F_{gm}(t,1) \geq_{st} F_{gm}(t,2).$$

Corollary (4.5) *Consider the family of SF's* $\{\overline{F}(t\,|\,\gamma), \gamma \in [0,\infty)\}$, *in which the baseline SF follows from the proportional reversed hazard rate model,* $\overline{F}(t\,|\,\gamma) = 1 - F^{\gamma}(t)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i, $i = 1,2$, *represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i) = \exp\left(\int_0^\infty \log(1 - F^\gamma(t))d\Pi_i(\gamma)\right).$$

If $\Pi_1 \leq_{st} \Pi_2$, then

$$F_{gm}(t,1) \leq_{st} F_{gm}(t,2).$$

Corollary (4.6) *Consider the family of SF's* $\{\overline{F}(t \mid \gamma), \gamma \in [0,\infty)\}$*, in which the baseline SF follows from the accelerated model,* $\overline{F}(t \mid \gamma) = \overline{F}(\gamma t)$*. Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2*, respectively. Furthermore, let* $tr(t)$ *be increasing in* t*, where* $r(t)$ *is the hazard rate of* $\overline{F}(t)$*. Also, suppose that* T_i*,* $i = 1,2$*, represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i) = \exp\left(\int_0^\infty \log(\overline{F}(\gamma t))d\Pi_i(\gamma)\right).$$

If $\Pi_1 \leq_{st} \Pi_2$, then

$$F_{gm}(t,1) \geq_{st} F_{gm}(t,2).$$

Remark (4.7) At first glance, the assumption $tr(t)$ presented in Corollary 4.6 may seem a very strong and restrictive assumption, and it is not possible to find a statistical distribution that follows the presented assumption in the corollary, while this is not true. The following example, which is a very important and well-known distribution in reliability theory, is satisfied the expressed assumption.

Example (4.8) Let the baseline SF be a Weibull distribution, $\overline{F}(t \mid \gamma) = \exp(-t^\gamma)$. It is easy to see that $r(t \mid \gamma) = \gamma t^{\gamma-1}$, and thus $tr(t) = \gamma t^\gamma$ is an increasing function of $t > 0$.

The following theorem is similar to Theorem 4.1 in the geometric mixture of CDF's.

Theorem (4.9) *Consider the family of CDF's* $\{F(t \mid \gamma), \gamma \in [0,\infty)\}$*. Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2*, respectively. Also, suppose that* T_i^**,* $i = 1,2$*, represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(F(t \mid \gamma))d\Pi_i(\gamma)\right).$$

If $F(t \mid \gamma) \leq_{st} F(t \mid \gamma')$ for $\gamma \leq \gamma'$ and $\Pi_1 \leq_{st} \Pi_2$, then

$$F_{gmC}(t,1) \leq_{st} F_{gmC}(t,2).$$

Proof. The proof of the theorem follows from the proof of Theorem 4.1.

The following corollaries can be obtained directly from Theorem 4.9 in the proportional hazard rate, the additive hazard, the proportional reversed hazard rate and accelerated lifetime models.

Corollary (4.10) *Consider the family of CDF's* $\{F(t \mid \gamma), \gamma \in [0,\infty)\}$*, in which the baseline CDF follows from the proportional hazard rate model,* $F(t \mid \gamma) = 1 - \overline{F}^\gamma(t)$*. Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2*, respectively. Also, suppose that* T_i^**,* $i = 1,2$*, represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(1 - \overline{F}^\gamma(t))d\Pi_i(\gamma)\right).$$

If $\Pi_1 \le_{st} \Pi_2$, then

$$F_{gmC}(t,1) \ge_{st} F_{gmC}(t,2).$$

Corollary (4.11) *Consider the family of CDF's* $\{F(t\mid\gamma), \gamma \in [0,\infty)\}$, *in which the baseline CDF follows from the additive hazard rate model,* $F(t\mid\gamma) = 1-\overline{F}(t)e^{-\gamma t}$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i^*, $i=1,2$, *represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(1-\overline{F}(t)e^{-\gamma t})d\Pi_i(\gamma)\right).$$

If $\Pi_1 \le_{st} \Pi_2$, then

$$F_{gmC}(t,1) \ge_{st} F_{gmC}(t,2).$$

Corollary (4.12) *Consider the family of SF's* $\{F(t\mid\gamma), \gamma \in [0,\infty)\}$, *in which the baseline CDF follows from the proportional reversed hazard rate model,* $F(t\mid\gamma) = F^{\gamma}(t)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i^*, $i=1,2$, *represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(F^{\gamma}(t))d\Pi_i(\gamma)\right).$$

If $\Pi_1 \le_{st} \Pi_2$, then

$$F_{gmC}(t,1) \le_{st} F_{gmC}(t,2).$$

Corollary (4.13) *Consider the family of CDF's* $\{F(t\mid\gamma), \gamma \in [0,\infty)\}$, *in which the baseline CDF follows from the accelerated model,* $F(t\mid\gamma) = 1-\overline{F}(\gamma t)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Furthermore, let* $tr(t)$ *be increasing in* t, *where* $r(t)$ *is the hazard rate of* $\overline{F}(t)$. *Also, suppose that* T_i^*, $i=1,2$, *represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(1-\overline{F}(\gamma t))d\Pi_i(\gamma)\right).$$

If $\Pi_1 \le_{st} \Pi_2$, then

$$F_{gmC}(t,1) \ge_{st} F_{gmC}(t,2).$$

4.2 *The hazard rate order*

In the following theorem, we extend the result of Theorem 1.B.14 of Shaked and Shanthikumar [12] in ordinary mixture to the geometric mixture of SF's. In fact, we compare two geometric mixtures of SF's in the sense of the hazard rate order.

Theorem (4.14) *Consider the family of SF's* $\{\overline{F}(t\mid\gamma), \gamma \in [0,\infty)\}$ *with CDF* $F(t\mid\gamma)$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i, $i=1,2$, *represent the random variables of geometric mixtures of SF's with the following SF's:*

$$\overline{F}_{gm}(t,i) = \exp\left(\int_0^\infty \log(\overline{F}(t \mid \gamma)) d\Pi_i(\gamma)\right).$$

If $F(t \mid \gamma) \leq_{hr} F(t \mid \gamma')$ for $\gamma \leq \gamma'$, and $\Pi_1 \leq_{hr} \Pi_2$, then

$$F_{gm}(t,1) \leq_{hr} F_{gm}(t,2)$$

where $F_{gm}(t,i)$, $i = 1,2$, is corresponding CDF of $\overline{F}_{gm}(t,i)$.

Proof. From assumption $F(t \mid \gamma) \leq_{hr} F(t \mid \gamma')$, $\overline{F}(t \mid \gamma)$ is a TP_2 function of $\gamma \in [0,\infty)$ and $t \in R$. Also, from $\Pi_1 \leq_{hr} \Pi_2$, $\overline{\Pi}_i(\gamma)$ as a function of $\gamma \in [0,\infty)$ and $i \in \{1,2\}$, is TP_2. Furthermore, $\overline{F}(t \mid \gamma)$ is increasing in γ (because $F \leq_{hr} G \Rightarrow F \leq_{st} G$), and then $\log(\overline{F}(t \mid \gamma))$ is, also, incresing in γ. Thus, from Theorem 2.3, we obtain

$$\int_0^\infty \log(\overline{F}(t \mid \gamma)) d\Pi_1(\gamma) \int_0^\infty \log(\overline{F}(t' \mid \gamma)) d\Pi_2(\gamma) \geq \int_0^\infty \log(\overline{F}(t \mid \gamma)) d\Pi_2(\gamma) \int_0^\infty \log(\overline{F}(t' \mid \gamma)) d\Pi_1(\gamma).$$

Now, by raising both sides of the above inequality to base e, we will have

$$\exp(\int_0^\infty \log(\overline{F}(t \mid \gamma)) d\Pi_1(\gamma)) \exp(\int_0^\infty \log(\overline{F}(t' \mid \gamma)) d\Pi_2(\gamma))$$

$$\geq \exp(\int_0^\infty \log(\overline{F}(t \mid \gamma)) d\Pi_2(\gamma)) \exp(\int_0^\infty \log(\overline{F}(t' \mid \gamma)) d\Pi_1(\gamma)).$$

Consequently,

$$\overline{F}_{gm}(t,1)\overline{F}_{gm}(t',2) \geq \overline{F}_{gm}(t,2)\overline{F}_{gm}(t',1),$$

and hence $\overline{F}_{gm}(t,i)$ is TP_2 in $\{1,2\}$ and $t \in R$. That means

$$F_{gm}(t,1) \leq_{hr} F_{gm}(t,2).$$

This complete the proof.

The following theorem is concerned with the reversed hazard rate order of the geometric mixture of CDF's, and extends the result of Theorem 1.B.52 of Shaked and Shanthikumar [12] to the model.

Theorem (4.15) *Consider the family of CDF's* $\{F(t \mid \gamma), \gamma \in [0,\infty)\}$. *Let the random variables* Γ_1 *and* Γ_2 *have CDF's* Π_1 *and* Π_2, *respectively. Also, suppose that* T_i^*, $i = 1,2$, *represent the random variables of geometric mixtures of CDF's with the following CDF's:*

$$F_{gmC}(t,i) = \exp\left(\int_0^\infty \log(F(t \mid \gamma)) d\Pi_i(\gamma)\right).$$

If $F(t \mid \gamma) \leq_{rh} F(t \mid \gamma')$ for $\gamma \leq \gamma'$, and $\Pi_1 \leq_{rh} \Pi_2$, then

$$F_{gmC}(t,1) \leq_{rh} F_{gmC}(t,2).$$

Proof. The proof of the theorem follows from the proof of Theorem 4.14.

5. Conclusion

In this paper, we have investigated a different mixture model to examine heterogeneity in populations. The geometric mixture model is the mixture failure rate (reversed failure rate) model which has been defined through a linear combination of the failure rate (reversed failure rate) of each component. we have focused on the mixture failure rate (reversed failure rate) model, and have investigated ageing properties as well as stochastic comparisons of the model.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding this article.

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