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On $N(k)$-quasi Einstein Manifolds Satisfying Some Conditions

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Abstract

In this paper, we study W_8-curvature tensor on $N(k)$-quasi Einstein manifolds. The tensor W_8 is defined as a modification of the Riemannian curvature tensor involving the Ricci operator. Several of its basic properties are first derived with respect to the structure vector field ξ, the associated 1-form η, and the Riemannian metric g. Using these relations, we investigate curvature conditions involving W_8. In particular, we consider the condition $R(\xi.X)\cdot W_8=0$ and $W_8(\xi.X)\cdot S=0$. All the results obtained are in the form of necessary and sufficient conditions.

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1. Introduction

The notion of a quasi Einstein manifold was introduced by Chaki in [1]. An n-dimensional Riemannian manifold M is said to be a quasi Einstein manifold if its Ricci tensor S satisfies

$$S(X.Y)=ag(X.Y)+b\eta(X)\eta(Y).\quad \forall X.Y\in TM \tag{1}$$

for some smooth functions a and $b\neq 0$, where η is a non zero 1-form such that

$$g(X.\xi)=\eta(X).\quad g(\xi.\xi)=\eta(\xi)=1. \tag{2}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If $b=0$, then the manifold reduces to an Einstein manifold. For more details about quasi Einstein manifolds see also [2, 6].

In [17], it was shown that a conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. The derivation conditions $R(\xi.X)\cdot R=0$ and $R(\xi.X)\cdot S=0$ were also studied in [17], where R and S denote the curvature and Ricci tensor, respectively. In [10], the derivation conditions $R(\xi.X)\cdot\rho=0, \rho(\xi.X)\cdot S=0$, and $\rho(\xi.X)\cdot\rho=0$ were studied where ρ is the projective curvature tensor. Also physical examples of $N(k)$-quasi Einstein manifolds were given. The derivation conditions $R(\xi.X)\cdot C=0$ and $R(\xi.X)\cdot\tilde{C}=0$ were studied in [11], where C and $\tilde{C}$ denote the conformal curvature tensor and quasi conformal curvature tensor, respectively. In [15], $N(k)$-quasi

Einstein manifolds satisfying the conditions $R(\xi.X)\cdot H=0, H(\xi.X)\cdot S=0, P(\xi.X)\cdot H=0, R(\xi.X)\cdot \bar{P}=0, and\ \bar{P}(\xi.X)\cdot S=0$, where H, P, and $\bar{P}$ denote the conharmonic curvature tensor, the projective curvature tensor and the pseudo projective curvature tensor, respectively, have been studied. In this paper, we consider N(k)-quasi Einstein manifolds satisfying the conditions $R(\xi.X)\cdot W_8=0$ and $W_8(\xi.X)\cdot S=0$. Chaubey, Pokhariyal and Siddiqi [1] studied curvature-restricted N(k)-quasi Einstein manifolds and obtained necessary and sufficient conditions for conformally and quasi-conformally flat cases, together with properties of the characteristic vector field. Hazra and Sarkar [2] examined N(k)-quasi Einstein manifolds with respect to the quasi-conformal curvature tensor and established related curvature identities along with illustrative examples.

2. $N(k)$-quasi Einstein manifolds

From (1) and (2) we obtain

$$S(X.\xi) = (a+b)\eta(X).$$
$$r = na+b$$

where r is the scalar curvature of M. The Ricci operator Q of a Riemannian manifold M is defined by

$$S(X.Y)=g(QX.Y).$$

If M is a quasi Einstein manifold [1], its Ricci operator satisfies

$$Q=aI+b\eta\otimes\xi.$$

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [16] of a Riemannian manifold defined by

$$N(k): p\longrightarrow N_v(k)=\{Z\in T_vM \mid R(X.Y)Z=k\{g(Y.Z)X-g(X.Z)Y\}\}$$

for all $X, Y\in TM^n$, where k is a smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then it is said to be an $N(k)$-quasi Einstein manifold [17].

Lemma 1 *[12] In an n-dimensional $N(k)$-quasi Einstein manifold, it follows that*

$$k=\frac{a+b}{n-1}.$$

Let M be an $N(k)$-quasi Einstein manifold. Then we have [12]

$$R(Y.Z)\xi=\frac{a+b}{n-1}\{\eta(Z)Y-\eta(Y)Z\}. \tag{3}$$

The equation (3) is equivalent to

$$R(\xi.Y)Z=\frac{a+b}{n-1}\{g(Y.Z)\xi-\eta(Z)Y\}=-R(Y.\xi)Z. \tag{4}$$

In [10], the following physical examples of $N(k)$-quasi Einstein manifolds are presented. In [17], Tripathi and Kim proved that an n-dimensional conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. Now we consider a conformally flat perfect fluid spacetime $(M^4.g)$ satisfying Einstein's equation without cosmological constant. Furthermore, let ξ be the unit time-like velocity vector of the fluid. It is known [9] that Einstein's equation without a cosmological constant can be written as

$$S(X.Y)-\frac{1}{2}rg(X.Y)=\kappa T(X.Y),$$

where κ is the gravitational constant and T is the energy momentum tensor of type (0.2). In the present case, this can be written as follows:

$$S(X.Y)-\frac{1}{2}rg(X.Y)=\kappa[(\sigma+p)\eta(X)\eta(Y)+pg(X.Y)],$$

where σ is the energy density and p is the isotropic pressure of the fluid. Then we have

$$S(X.Y) = \left(\kappa p + \frac{1}{2}r\right) g(X.Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

Since the spacetime is conformally flat, by [15], it is $N(k)$-quasi Einstein. From the above equation, by a contraction we get

$$r = \kappa(\sigma - 3p).$$

Hence, $S(X.Y)$ can be written as

$$S(X.Y) = \left(\frac{\kappa}{2}(\sigma - p)\right) g(X.Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

Therefore, from (1) we have

$$a = \frac{\kappa}{2}(\sigma - p)$$

and

$$b = \kappa(\sigma + p).$$

Hence, we can state the following example.

Example 1 *[10] A conformally flat perfect fluid spacetime ($M^4.g$) satisfying Einstein's equation without cosmological constant is an $N(\kappa(3\sigma + p)/6)$-quasi Einstein manifold.*

3. W_8-curvature tensor of an $N(k)$-quasi Einstein manifold

Let M be a Riemannian manifold. The W_8 curvature tensor [13] is defined by

$$W_8(X.Y)Z = R(X.Y)Z - \frac{1}{n-2}\{S(Y.Z)X - S(X.Y)Z\}.$$

Proposition 1 *In an n-dimensional $N(k)$-quasi Einstein manifold M, the curvature tensor W_8 satisfies*

$$W_8(\xi.Y)Z = \frac{a+b}{n-1}\{\eta(Y)Z - \eta(Z)Y\} + \frac{b}{n-1}\{g(Y.Z)\xi - \eta(Y)\eta(Z)\xi\}, \tag{5}$$

$$W_8(X.\xi)Z = \frac{a+b}{n-1}\{\eta(X)Z - g(X.Z)\xi\}, \tag{6}$$

$$W_8(X.Y)\xi = \frac{1}{n-1}\{ag(X.Y)\xi + b\eta(X)\eta(Y)\xi - (a+b)\eta(X)Y\}, \tag{7}$$

$$\eta(W_8(X.Y)Z) = \frac{1}{n-1}\{ag(X.Y)\eta(Z) + bg(Y.Z)\eta(X) - (a+b)g(X.Z)\eta(Y)\}. \tag{8}$$

Proof. From (1)-(4), equations (5)-(8) follow directly.

Also, $R \cdot W_8$ is defined by

$$(R(\xi.X) \cdot W_8)(Y.Z.U) = R(\xi.X)W_8(Y.Z)U - W_8(R(\xi.X)Y.Z)U \tag{9}$$
$$-W_8(Y.R(\xi.X)Z)U - W_8(Y.Z)R(\xi.X)U$$

where R denote the Riemannian curvature tensor of a Riemannian manifold M [8]. Now, we prove the following theorem:

Theorem 1 *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $R(\xi.X) \cdot W_8 = 0$ if and only if $a + b = 0$.*

Proof. Let M be an $N(k)$-quasi Einstein manifold and satisfies the condition $R(\xi.X) \cdot W_8 = 0$, then from (9) we can write

$$0 = R(\xi.X)W_8(Y.Z)U - W_8(R(\xi.X)Y.Z)U \tag{10}$$
$$-W_8(Y.R(\xi.X)Z)U - W_8(Y.Z)R(\xi.X)U$$

for all vector fields $X.Y.Z.U$ on M. So from (4) in (10) we obtain

$$0 = \frac{a+b}{n-1}\{W_8(Y.Z.U.X)\xi - \eta(W_8(Y.Z)U)X$$
$$-g(X.Y)W_8(\xi.Z)U + \eta(Y)W_8(X.Z)U$$
$$-g(X.Z)W_8(Y.\xi)U + \eta(Z)W_8(Y.X)U$$
$$-g(X.U)W_8(Y.Z)\xi + \eta(U)W_8(Y.Z)X\}.$$

which implies either $a + b = 0$ or

$$0 = W_8(Y.Z.U.X)\xi - \eta(W_8(Y.Z)U)X \tag{11}$$
$$-g(X.Y)W_8(\xi.Z)U + \eta(Y)W_8(X.Z)U$$
$$-g(X.Z)W_8(Y.\xi)U + \eta(Z)W_8(Y.X)U$$
$$-g(X.U)W_8(Y.Z)\xi + \eta(U)W_8(Y.Z)X.$$

holds on M. Taking the inner product of both sides of (11) with ξ we obtain

$$0 = W_8(Y.Z.U.X) - \eta(W_8(Y.Z)U)\eta(X) \tag{12}$$
$$-g(X.Y)\eta(W_8(\xi.Z)U) + \eta(Y)\eta(W_8(X.Z)U)$$
$$-g(X.Z)\eta(W_8(Y.\xi)U) + \eta(Z)\eta(W_8(Y.X)U)$$
$$-g(X.U)\eta(W_8(Y.Z)\xi) + \eta(U)\eta(W_8(Y.Z)X).$$

Putting (5)-(8) in (12) we obtain

$$0 = W_8(Y.Z.U.X) - \frac{a}{n-1}\{g(X.U)g(Z.Y) - g(X.Z)g(Y.U)\}$$
$$-\frac{b}{n-1}\{g(X.Y)g(Z.U) - g(X.Z)g(Y.U)\}.$$

Hence we have

$$W_8(Y.Z.U.X) = \frac{a}{n-1}\{g(X.U)g(Z.Y) - g(X.Z)g(Y.U)\}$$
$$+\frac{b}{n-1}\{g(X.Y)g(Z.U) - g(X.Z)g(Y.U)\}. \tag{13}$$

Setting $X = Y = e_i$ in (13), where $\{e_i : i = 1.2.\cdots.n\}$ is an orthonormal basis of the tangent space at any point, and summing over i yields

$$0 = \frac{b}{n-1}g(Z.U).$$

The above equation implies that $b = 0$, but since M is an $N(k)$-quasi Einstein manifold, then its impossible. The converse statement is trivial. This completes the proof of the theorem.

Next, we have the following theorem

Theorem 2 *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $W_8(\xi.X)\cdot S = 0$ if and only if $a + b = 0$.*

Proof. Since $W_8(\xi.X)\cdot S = 0$, we have

$$S(W_8(\xi.X)Y.Z) + S(Y.W_8(\xi.X)Z) = 0 \tag{14}$$

In view of (1.1) in (3.7) we have

$$(W_8(\xi.X)Y.Z) = \frac{b(a+b)}{n-1}\{(g(X.Y) - \eta(X)\eta(Y))\eta(Z)\} \tag{15}$$
$$+\frac{a(a+b)}{n-1}\{g(Y.Z)\eta(X) - g(X.Z)\eta(Y)\}.$$

and

$$S(Y.W_8(\xi.X)Z) = \frac{b(a+b)}{n-1}\{(g(X.Z) - \eta(X)\eta(Z))\eta(Y)\} \tag{16}$$
$$+\frac{a(a+b)}{n-1}\{g(Y.Z)\eta(X) - g(X.Y)\eta(Z)\}.$$

In view of (15) and (16) in (14) we have

$$0 = \frac{b(a+b)}{n-1}\{g(X.Z)\eta(Y) + g(X.Y)\eta(Z) - 2\eta(X)\eta(Z)\eta(Y)\} \quad (17)$$
$$+\frac{a(a+b)}{n-1}\{2g(Y.Z)\eta(X) - g(X.Y)\eta(Z) - g(X.Z)\eta(Y)\}$$
$$= \frac{(a+b)}{n-1}\{bg(X.Z)\eta(Y) + bg(X.Y)\eta(Z) - 2b\eta(X)\eta(Z)\eta(Y)$$
$$+2ag(Y.Z)\eta(X) - ag(X.Y)\eta(Z) - ag(X.Z)\eta(Y)\}.$$

From (17) by a contraction, we obtain

$$\frac{a+b}{n-1} = 0.$$

which give us $a + b = 0$. The converse statement is trivial. This completes the proof of the theorem.

From Theorem1 and Theorem2, we can see the following:

Corollary 1 *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. The following relations are equivalent.*

(I) $a + b = 0$.

(II) $R(\xi.X) \cdot W_8 = 0$.

(III) $W_8(\xi.X) \cdot S = 0$.

4. Conclusion

In this paper, we have investigated the W_8-curvature tensor introduced by G. P. Pokhariyal on $N(k)$-quasi Einstein manifolds. After recalling the basic structure of quasi Einstein manifolds, originally introduced by M. C. Chaki, and the definition of $N(k)$-quasi Einstein manifolds studied by M. M. Tripathi and others, we derived several fundamental identities describing the interaction of the W_8-tensor with the structure vector field ξ, the associated 1-form η, and the Riemannian metric g.

Using these identities, we examined two derivation-type curvature conditions, namely

$R(\xi.X) \cdot W_8 = 0$ and $W_8(\xi.X) \cdot S = 0$.

It has been shown that each of these conditions is equivalent to the algebraic relation $a + b = 0$. Consequently, the three statements

$a + b = 0,\quad R(\xi.X) \cdot W_8 = 0,\quad W_8(\xi.X) \cdot S = 0.$

are mutually equivalent on an n-dimensional $N(k)$-quasi Einstein manifold.

Thus, the paper provides a complete characterization of $N(k)$-quasi Einstein manifolds satisfying the above curvature restrictions in terms of a simple condition on the structure functions a and b. These results contribute to the ongoing study of curvature-restricted geometric structures and may motivate further investigations of other generalized curvature tensors under similar derivation conditions.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding this article.

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